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The number of spanning trees of plane graphs with reflective symmetry☆

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Abstract

A plane graph is called symmetric if it is invariant under the reflection across some straight line (called symmetry axis). Let G be a symmetric plane graph. We prove that if there is no edge in G intersected by its symmetry axis then the number of spanning trees of G can be expressed in terms of the product of the number of spanning trees of two smaller graphs, each of which has about half the number of vertices of G.

MSC: 05C80

Keywords: Symmetric graph; Dual graph; Spanning tree; Matrix-Tree Theorem; Perfect matching

1. Introduction

Throughout this paper, we assume that $G = (V(G), E(G))$ is a connected and unweighted graph with no loops, having vertex set $V(G) = \{a_1, a_2, \ldots, a_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Denote the degree of vertex a_i by $d_G(a_i)$ (or $d(a_i)$), the diagonal matrix of vertex degrees of G by $D(G)$, the adjacency matrix of G by $A(G)$, and the Laplacian matrix of G by $L(G) = D(G) - A(G)$. The reader is referred to Biggs [1] for terminology and notation not defined here.

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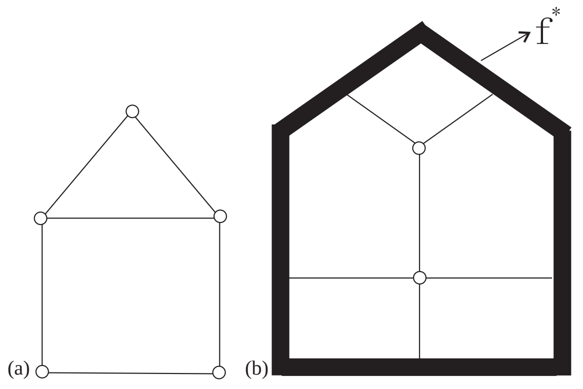

Fig. 1. (a) A plane graph G. (b) The dual graph of G.

Methods for enumerating spanning trees in a finite graph, a problem related to various areas of mathematics and physics, have been investigated for more than 150 years (see [10]). We denote the number of spanning trees of the graph G by $t(G)$. A well-known formula for $t(G)$ is "the Matrix-Tree Theorem" (see e.g. [1] or [2]), which expresses it as a determinant.

Theorem 1 (*Biggs [1], Bondy and Murty [2]; the Matrix-Tree Theorem*). *Let G be a graph with n vertices and denote by $L(G)$ the Laplacian matrix of G. Then $t(G)$, the number of spanning trees of G, equals the determinant of the submatrix obtained by deleting row a_r and column a_r from $L(G)$ for any $1 \leqslant r \leqslant n$.*

Given a plane graph G (see Fig. 1(a)), we denote the dual graph of G by $G^\perp$ (see Fig. 1(b)); its vertices, edges and faces correspond to faces, edges and vertices of G, respectively (including a vertex, here marked f^*, that corresponds to the unbounded, external face of G, and is represented in "extended form", i.e., as a spread-out region rather than a small dot). We can embed G and $G^\perp$ simultaneously in the plane, such that an edge e of G crosses the corresponding dual edge $e^\perp$ of $G^\perp$ exactly once and crosses no other edge of $G^\perp$. Given a spanning tree T of G, the edges of $G^\perp$ that do not cross edges of T form a spanning tree of $G^\perp$; this is called the dual tree and we denoted it by $T^\perp$. There is a standard bijection $T \longmapsto T^\perp$ between the spanning trees of G and those of $G^\perp$. Namely, if T has edge set $\{e_1, e_2, \dots, e_{n-1}\}$, then $T^\perp$ has edge set $E(G^\perp)\backslash\{e_1^\perp, e_2^\perp, \dots, e_{n-1}^\perp\}$, where $E(G^\perp)$ denotes the edge set of $G^\perp$. Hence we have the following

Theorem 2 (*Lóvasz [11], Stanley [14]*). *Suppose that G is a connected plane graph and $G^\perp$ is the dual graph of G. Then*

$$t(G) = t(G^\perp).$$

The above result can be found for instance in [14, exercise 5.72]; [11, §5, exercise 23]. Some related work appears in Refs. [3,4,7,9,12,13].

Remark 3. Suppose that G is a connected plane graph with weights on its edges. Let $t(G)$ denote the sum of the weights of the spanning trees of G, where the weight of a spanning tree T of G is the product of the weights of the edges of T. Let $G^{\perp}$ be the dual of G, with the weight of edge $e^{\perp}$ of $G^{\perp}$ taken to be the same as the weight of the corresponding edge e in G. Then it is easy to see that $t(G)$ and $t(G^{\perp})$ are not equal in general.

This paper is inspired by two results, one of which concerns the bijection between spanning trees of a general plane graph and perfect matchings of a related graph (see e.g. [15] or [11]). The second is the matching factorization theorem related to the number of perfect matchings of a class of graphs with reflective symmetry presented in [5]. The matching factorization theorem expresses the number of perfect matchings of a symmetric plane bipartite graph G in terms of the product of the number of perfect matchings of two subgraphs of G, each of which has about half the number of vertices of G. Based on this it is natural to ask whether there exists a similar result for the number of spanning trees of a plane graph with reflective symmetry. The main result of this paper, Theorem 4, answers this question in the affirmative. We present both an algebraic and a combinatorial solution for this.

The result stated in Theorem 4 was found by W. Yan, F. Zhang, who also gave the algebraic proof. The combinatorial proof was supplied by M. Ciucu.

2. Main result

Let G be a connected plane graph. We say that G is symmetric if it is invariant under the reflection across some straight line ℓ (called symmetry axis). We consider ℓ to be vertical. Fig. 2(a) shows an example of a symmetric graph. Let G be a symmetric plane graph with symmetry axis ℓ intersecting no edge of G (edges lying entirely along the symmetry axis are allowed, like for instance edges a_2a_3, a_4a_5 and a_5a_6 in the graph G showed in Fig. 2(a)). The number of bounded faces of G intersected by its symmetry axis is denoted by $\omega(G)$. For the graph G pictured in Fig. 2(a), there are two bounded faces, here marked f_1 and f_2, intersected by ℓ, so $\omega(G) = 2$. Let $a_1, a_2, \dots, a_k$ be the vertices of G lying on ℓ. Let G'_{L} and G'_{R} be the subgraphs of G at the left and right sides of ℓ, respectively. We denote the subgraphs of G induced by $V(G'_{\mathrm{L}}) \cup \{a_1, a_2, \dots, a_k\}$ and $V(G'_{\mathrm{R}}) \cup \{a_1, a_2, \dots, a_k\}$ by G_{L} and G_{R}, respectively. Let G_1 be the graph obtained from G_{L} by subdividing once each edge of G_{L} lying on the symmetry axis, and G_2 the graph obtained from G_{R} by identifying all vertices $a_1, a_2, \dots, a_k$ (any loops created by the identification of the vertices on ℓ are discarded). Figs. 3 and 4 illustrate this procedure for the graph pictured in Fig. 2(a). Now we can state our main result as follows.

Theorem 4. *Let G be a symmetric plane graph with symmetry axis ℓ intersecting no edge of G, and let G_1 and G_2 be the graphs defined above. Then the number of spanning trees of G is given by*

$$t(G) = 2^{\omega(G)} t(G_1) t(G_2),$$

where $\omega(G)$ denotes the number of bounded faces of G intersected by ℓ.

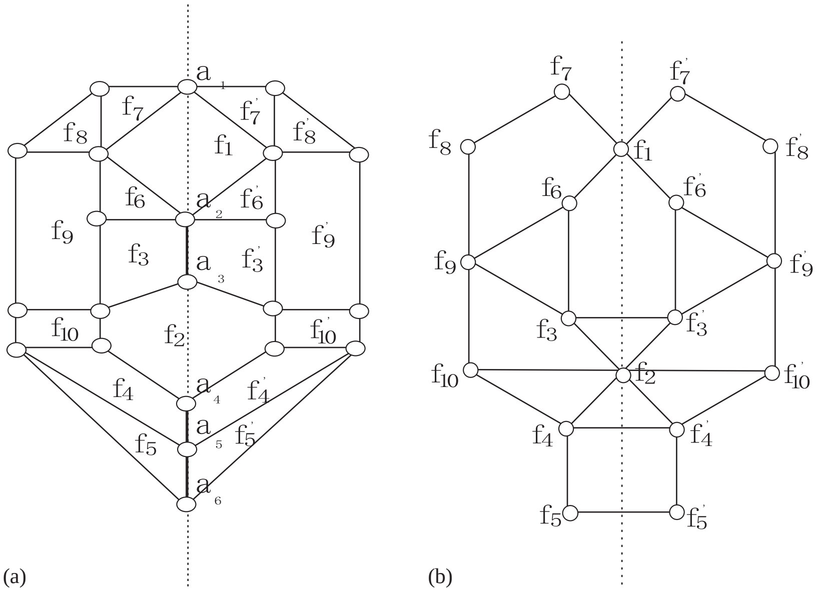

Fig. 2. (a) A symmetric plane graph G. (b) The graph $G^{\perp} - f^*$.

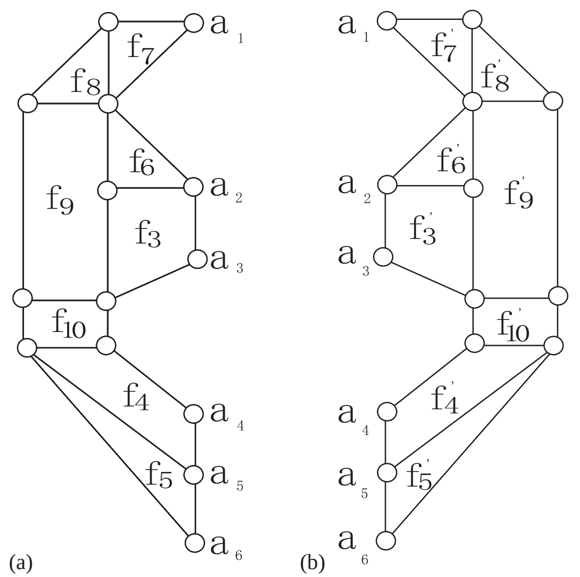

Fig. 3. The graph G_{L}. (b) The graph G_{R}.

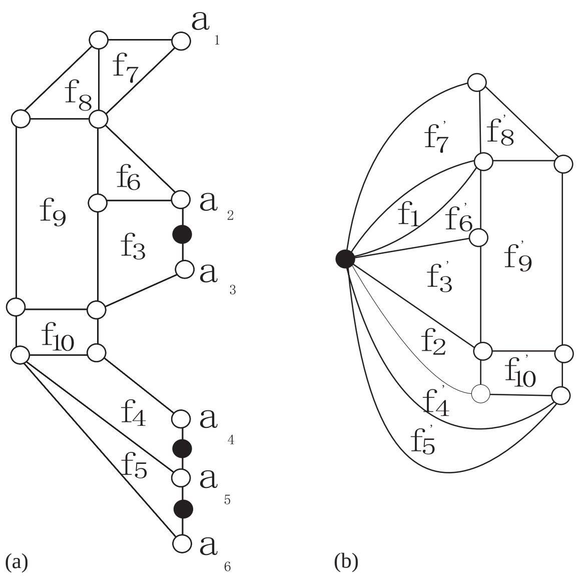

Fig. 4. (a) The graph G_1. (b) The graph G_2.

We will give two methods to prove Theorem 4, one algebraic and the other combinatorial.

Algebraic proof of Theorem 4. Without loss of generality, we may assume that G is connected. Let $G^{\perp}$ be the dual graph of G. Denote the bounded faces of G intersected by the symmetry axis by $f_1, f_2, \ldots, f_{\omega(G)}$, from top to bottom. It is not difficult to see that the following claims hold (see Fig. 2(b)).

Claim 1. *$G^{\perp} - f^*$ can be drawn as a symmetric plane graph with the same symmetry axis ℓ, where f^* is the vertex of $G^{\perp}$ corresponding to the unbounded, external face of G, and $G^{\perp} - f^*$ denotes the subgraph of $G^{\perp}$ induced by deleting vertex f^* from $G^{\perp}$.*

Claim 2. *There exist exactly $\omega(G)$ vertices of $G^{\perp} - f^*$ lying on the symmetry axis ℓ; denote them also by f_i $(i = 1, 2, \ldots, \omega(G))$. Moreover, the subgraph of $G^{\perp}$ induced by the vertices f_i for $i = 1, 2, \ldots, \omega(G)$ is an "even" weighted graph, that is, there are $2s_{ij}$ $(s_{ij} \geqslant 0)$ edges from vertex f_i to vertex f_j for $i, j = 1, 2, \ldots, \omega(G)$, where $2s_{ij}$ is the number of common edges of the faces f_i and f_j of G for $i = 1, 2, \ldots, \omega(G)$.*

Claim 3. *The edges of $G^{\perp} - f^*$ that cross the symmetry axis ℓ (if such edges exist) form a (partial) matching K of $G^{\perp} - f^*$. Moreover, the reflection across ℓ interchanges the endpoints of each edge of K. For the graph $G^{\perp} - f^*$ in Fig.* 2(b), *there exist three edges $f_3 f_3'$, $f_4 f_4'$ and $f_5 f_5'$ crossing the symmetry axis ℓ, which form a matching of $G^{\perp} - f^*$.*

Let $A(G^{\perp} - f^*)$ denote the adjacency matrix of $G^{\perp} - f^*$, and let A be the adjacency matrix of the graph $(G^{\perp} - f^*)'_{\mathrm{L}}$, which is the subgraph of $G^{\perp} - f^*$ induced by the vertices

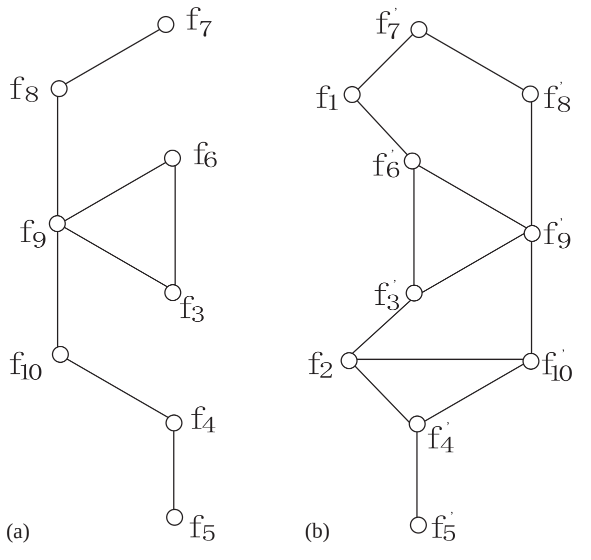

Fig. 5. (a) The graph $G_1^{\perp} - f_1^*$ (or $(G^{\perp} - f^*)'_{\mathrm{L}}$, or $G_{\mathrm{L}}^{\perp} - f_{\mathrm{L}}^*$). (b) The graph $G_2^{\perp} - f_2^*$.

of $G^{\perp} - f^*$ at the left side of ℓ (see Figs. 2(b) and 5(a)). By the definition of G_1, it is not difficult to see that $(G^{\perp} - f^*)'_{\mathrm{L}}$ and $G_1^{\perp} - f_1^*$ (or $G_{\mathrm{L}}^{\perp} - f_{\mathrm{L}}^*$) are isomorphic, where $G_1^{\perp}$ and $G_{\mathrm{L}}^{\perp}$ are the dual graphs of G_1 and G_{L}, respectively, and f_1^* and f_{L}^* are the vertices of $G_1^{\perp}$ and $G_{\mathrm{L}}^{\perp}$ corresponding to the unbounded, external faces of G_1 and G_{L}. Therefore the following claim holds.

Claim 4. *The adjacency matrix of $G_1^{\perp} - f_1^*$ is A.*

Suppose that matrix B denotes the incidence relations between the vertices of the graph $(G^{\perp} - f^*)'_{\mathrm{L}}$ and the vertices of $G^{\perp} - f^*$ lying on ℓ, and matrix C denotes the incidence relations between $(G^{\perp} - f^*)'_{\mathrm{L}}$ and $(G^{\perp} - f^*)'_{\mathrm{R}}$, which is the subgraph of $G^{\perp} - f^*$ induced by vertices of $G^{\perp} - f^*$ at the right side of ℓ. It is clear that $(G^{\perp} - f^*)'_{\mathrm{L}}$ and $(G^{\perp} - f^*)'_{\mathrm{R}}$ are two isomorphic subgraphs of $G^{\perp} - f^*$. Let $X = (x_{ij})_{\omega(G)\times\omega(G)}$ denote the adjacency matrix of the subgraph of $G^{\perp}$ induced by the vertices $f_1, f_2, \dots, f_{\omega(G)}$. If we list first the vertices of $G^{\perp} - f^*$ that are in $V((G^{\perp} - f^*)_{\mathrm{L}})$, then those lying on the symmetry axis ℓ, and finally those in $V((G^{\perp} - f^*)_{\mathrm{R}})$, then, by Claim 1, $A(G^{\perp} - f^*)$ has the following form:

$$A(G^{\perp} - f^*) = \begin{pmatrix} A & B & C \\ B^T & X & B^T \\ C^T & B & A \end{pmatrix}.$$

Note that, by Claim 3, C is represented by a diagonal matrix. Hence we have

$$A(G^{\perp} - f^{*}) = \begin{pmatrix} A & B & C \\ B^{T} & X & B^{T} \\ C & B & A \end{pmatrix},$$

where $\begin{pmatrix} X & B^{T} \\ B & A \end{pmatrix}$ is the adjacency matrix of the subgraph of $G^{\perp} - f^{*}$ induced by $V(G^{\perp} - f^{*})\backslash V((G^{\perp} - f^{*})'_{\mathrm{L}})$ (see Fig. 5(b)), denoted by $G[V(G^{\perp} - f^{*})\backslash V((G^{\perp} - f^{*})'_{\mathrm{L}})]$. Note that, by the definition of G_2, G_2 is obtained from G_{R} by identifying all vertices $a_1, a_2, \ldots, a_k$ lying on the symmetry axis ℓ. It is clear that there is a natural way to identify faces in G_2 and in G. By Claim 2, it is not difficult to see that if the number of common edges of f_i and f_j $(i, j = 1, 2, \ldots, \omega(G))$ in $G^{\perp} - f^{*}$ is $2s_{ij}$ then the number of common edges of f_i and f_j $(i, j = 1, 2, \ldots, \omega(G))$ in $G_2^{\perp} - f_2^{*}$ is s_{ij}, where f_2^{*} is the vertex of $G_2^{\perp}$ corresponding to the unbounded, external face of G_2. Hence the following claim holds.

Claim 5. *The adjacency matrix of* $G_2^{\perp} - f_2^{*}$ *is* $\begin{pmatrix} \frac{1}{2}X & B^{T} \\ B & A \end{pmatrix}$.

Let $D(G^{\perp})$ and $A(G^{\perp})$ denote the diagonal matrix of vertex degrees and the adjacency matrix of $G^{\perp}$, respectively. Then the submatrix of the Laplacian $L(G^{\perp})$ of $G^{\perp}$ obtained by deleting row f^{*} and column f^{*} from $L(G^{\perp})$ has the following form:

$$\begin{pmatrix} D_1 & & \\ & D_2 & \\ & & D_1 \end{pmatrix} - \begin{pmatrix} A & B & C \\ B^{T} & X & B^{T} \\ C & B & A \end{pmatrix} = \begin{pmatrix} D_1 - A & -B & -C \\ -B^{T} & D_2 - X & -B^{T} \\ -C & -B & D_1 - A \end{pmatrix},$$

where D_1 is the diagonal submatrix of $D(G^{\perp})$ corresponding to those vertices of $G^{\perp} - f^{*}$ on the left side of ℓ, and D_2 is the diagonal submatrix of $D(G^{\perp})$ corresponding to those vertices of $G^{\perp} - f^{*}$ lying on ℓ. For the graph G in Fig. 2(a), the vertices f_i $(3 \leqslant i \leqslant 10)$ of $G^{\perp}$ are on the left side of ℓ and the vertices f_1 and f_2 are on the symmetry axis. Thus, the entries of the diagonal submatrix D_1 are $d_{G^{\perp}}(f_i)$ $(3 \leqslant i \leqslant 10)$ and the entries of the diagonal submatrix D_2 are $d_{G^{\perp}}(f_1)$ and $d_{G^{\perp}}(f_2)$; by Fig. 2(a), $d_{G^{\perp}}(f_1) = 4, d_{G^{\perp}}(f_2) = 6, d_{G^{\perp}}(f_3) = 4, d_{G^{\perp}}(f_4) = 4, d_{G^{\perp}}(f_5) = 3, d_{G^{\perp}}(f_6) = 3, d_{G^{\perp}}(f_7) = 3, d_{G^{\perp}}(f_8) = 3, d_{G^{\perp}}(f_9) = 5$, and $d_{G^{\perp}}(f_{10}) = 4$. Hence, for the graph G showed in Fig. 2(a), the corresponding matrices A, B, C, X, D_1 and D_2 are:

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$C = \mathrm{diag}\,(1, 1, 1, 0, 0, 0, 0, 0), \quad D_1 = \mathrm{diag}\,(4, 4, 3, 3, 3, 3, 5, 4), \quad X = 0_2, \quad D_2 = \mathrm{diag}\,(4, 6),$$

where 0_2 is a 2×2 zero matrix.

Therefore, by the Matrix-Tree Theorem (Theorem 1) and Theorem 2, we have

$$\begin{aligned}
t(G) = t(G^{\perp}) &= \det\begin{pmatrix} D_1 - A & -B & -C \\ -B^T & D_2 - X & -B^T \\ -C & -B & D_1 - A \end{pmatrix} \\
&= \det\begin{pmatrix} D_1 - A & -B & -C \\ -B^T & D_2 - X & -B^T \\ D_1 - A - C & -2B & D_1 - A - C \end{pmatrix} \\
&= \det\begin{pmatrix} D_1 + C - A & -B & -C \\ 0 & D_2 - X & -B^T \\ 0 & -2B & D_1 - C - A \end{pmatrix} \\
&= \det(D_1 + C - A)\det\begin{pmatrix} D_2 - X & -B^T \\ -2B & D_1 - C - A \end{pmatrix}.
\end{aligned}$$

Note that D_2 is an $\omega(G) \times \omega(G)$ matrix, hence we have

$$\begin{aligned}
t(G) = t(G^{\perp}) &= \det(D_1 + C - A)\det\begin{pmatrix} D_2 - X & -B^T \\ -2B & D_1 - C - A \end{pmatrix} \\
&= 2^{\omega(G)}\det(D_1 + C - A)\det\begin{pmatrix} \frac{1}{2}(D_2 - X) & -B^T \\ -B & D_1 - C - A \end{pmatrix}.
\end{aligned}$$

Therefore, in order to prove Theorem 4, it suffices to prove that the following two equalities hold:

$$t(G_1) = \det(D_1 + C - A), \quad t(G_2) = \det\begin{pmatrix} \frac{1}{2}(D_2 - X) & -B^T \\ -B & D_1 - C - A \end{pmatrix}.$$

Note that $t(G_1) = t(G_1^{\perp})$ and $t(G_2) = t(G_2^{\perp})$. Thus, by Claims 4 and 5, it is enough to prove that the following two claims hold.

Claim 6. *Matrix $D_1 + C$ is the diagonal submatrix of $D(G_1^{\perp})$ obtained from the diagonal matrix $D(G_1^{\perp})$ of vertex degrees of $G_1^{\perp}$ by deleting row f_1^* and column f_1^*.*

Claim 7. *Matrix $\begin{pmatrix} \frac{1}{2}D_2 & 0 \\ 0 & D_1 - C \end{pmatrix}$ is the diagonal submatrix of $D(G_2^{\perp})$ obtained from the diagonal matrix $D(G_2^{\perp})$ of vertex degrees of $G_2^{\perp}$ by deleting row f_2^* and column f_2^*.*

First, we prove Claim 6. Since C, which is a diagonal matrix, denotes the incidence relations between vertices of $(G^{\perp} - f^*)'_{\mathrm{L}}$ and those of $(G^{\perp} - f^*)'_{\mathrm{R}}$, the (i,i)-entry of $D_1 + C$ equals $d_{G^{\perp}}(f_i) + c_{ii}$, where $d_{G^{\perp}}(f_i)$ is the degree of vertex f_i of $G^{\perp}$ (i.e., the number $d_G(f_i)$ of edges on the boundary of the face f_i of G), and

$$c_{ii} = \begin{cases} 1 & \text{if there exists an edge on the boundary of the} \\ & \text{face } f_i \text{ of } G \text{ lying on the symmetry axis } l, \\ 0 & \text{otherwise.} \end{cases}$$

Set

$$\Phi_{\mathrm{L}} = \{f \mid f \text{ is a bounded face of } G \text{ which is at the left side of the symmetry axis } l\},$$
$$\Phi_{\mathrm{R}} = \{f \mid f \text{ is a bounded face of } G \text{ which is at the right side of the symmetry axis } l\},$$
$$\Phi_{\mathrm{M}} = \{f \mid f \text{ is a bounded face of } G \text{ which is intersected by the symmetry axis } l\}.$$

It is clear that $V(G^{\perp}) = \Phi_{\mathrm{L}} \cup \Phi_{\mathrm{M}} \cup \Phi_{\mathrm{R}} \cup \{f^*\}$.

Note that, by the definition of G_1, G_1 is obtained from G_{L} by subdividing once every edge lying on the symmetry axis ℓ. Hence, for every face $f_i \in \Phi_{\mathrm{L}}$ of G on the left side of the symmetry axis (which may correspond to a face f_i in G_1), if there is an edge on the boundary of the face f_i lying on the symmetry axis ℓ, then $d_{G_1}(f_i) = d_G(f_i) + 1$; otherwise, $d_{G_1}(f_i) = d_G(f_i)$. So we have proved that $D_1 + C$ is the diagonal submatrix of $D(G_1^{\perp})$ obtained from the diagonal matrix $D(G_1^{\perp})$ of vertex degrees of $G_1^{\perp}$ by deleting row f_1^* and column f_1^*. This proves Claim 6.

Now we turn to proving Claim 7. Note that, by the definition of G_2, G_2 is obtained from G_{R} by identifying all vertices $a_1, a_2, \ldots, a_k$ lying on the symmetry axis ℓ. Hence, for every face $f_i \in \Phi_{\mathrm{M}} \cup \Phi_{\mathrm{R}}$ of G (which may corresponds to a face f_i in G_2), if $f_i \in \Phi_{\mathrm{M}}$ we have $d_{G_2}(f_i) = \frac{1}{2} d_G(f_i)$. For $f_i \in \Phi_{\mathrm{R}}$, if there is an edge on the boundary of the face f_i lying on the symmetry axis ℓ then $d_{G_2}(f_i) = d_G(f_i) - 1$; otherwise, $d_{G_2}(f_i) = d_G(f_i)$. So we have showed that $\begin{pmatrix} \frac{1}{2}D_2 & 0 \\ 0 & D_1 - C \end{pmatrix}$ is the diagonal submatrix of $D(G_2^{\perp})$ obtained from the diagonal matrix $D(G_2^{\perp})$ consisting of the vertex degrees of $G_2^{\perp}$ by deleting row f_2^* and column f_2^*. This proves Claim 7, and concludes our first proof of Theorem 4.

Before presenting the combinatorial proof of Theorem 4, we need to state in detail the connection between spanning tree and perfect matching enumeration mentioned in the Introduction. This is given by Lemma 5.

For a weighted graph G, the weight of a spanning tree is defined to be the product of the weights of all the edges of the spanning tree, and $t(G)$ as the sum the weights of all the spanning trees of G. Similarly, the weight of a perfect matching is the product of the weights of the edges in it. Let $M(G)$ denote the sum of the weights of all perfect matchings of G.

Lemma 5 (*Temperley [5], Lovász [14, Exercise 4.30]*). *Let G be a weighted plane graph with vertex set $V = \{a_1, \ldots, a_n\}$ and edge set $E = \{e_1, \ldots, e_m\}$. Let $\{f_1, \ldots, f_p\}$ be the bounded faces of G. Choose b_i to be a point in the interior of the edge e_i, and c_j a point in the interior of the face f_j, for $i = 1, \ldots, m$, $j = 1, \ldots, p$.*

Define $T(G)$ to be the weighted graph with vertex set $\{a_1, \ldots, a_n, b_1, \ldots, b_m, c_1, \ldots, c_p\}$ obtained by including all edges of the following two types (see Figs. 6(a) and (b) for an illustration):

(i) if b_i is on edge $\{a_k, a_l\}$ of G, include $\{b_i, a_k\}$ and $\{b_i, a_l\}$ as edges of $T(G)$; give each of them the weight of $\{a_k, a_l\}$;

(ii) if c_j is in the interior of a face bounding k edges, and the b-type vertices around this face are $\{b_{q_1}, \ldots, b_{q_k}\}$, include edges $\{c_j, b_{q_1}\}, \ldots, \{c_j, b_{q_k}\}$ as edges of $T(G)$, and weight them by 1.

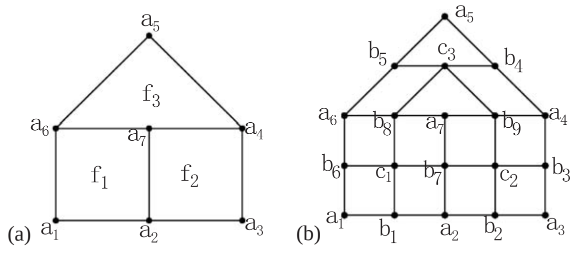

Fig. 6. (a) A plane graph G. (b) The graph $T(G)$.

Let $v \in \{a_1, \ldots, a_n\}$ be a vertex on the unbounded face of $T(G)$. Then

$$t(G) = M(T(G)\backslash v).$$

Edges whose weight we do not indicate explicitly are considered to have weight 1. If all weights are 1, $t(G)$ and $M(G)$ become the number of spanning trees and the number of perfect matchings of G, respectively.

Combinatorial proof of Theorem 4. Let v be the topmost vertex of G on the symmetry axis ℓ. Denote the graph $T(G)\backslash v$ by H. Then by the above lemma we have $t(G) = M(H)$.

Clearly, H can be drawn in the plane so as to be symmetric about the symmetry axis ℓ. In addition, each edge of $T(G)$ has one endpoint in $\{a_1, \ldots, a_n, c_1, \ldots, c_p\}$ and the other in $\{b_1, \ldots, b_m\}$, so H is bipartite. Thus we can apply to it the factorization theorem for perfect matchings presented in [6].

Let P be a plane curve that closely approximates ℓ and leaves all the a- and b-type vertices on ℓ on the left, and all c-type vertices on ℓ on the right. It follows then from the factorization theorem of [6] that

$$M(H) = 2^{w(H)} M(H_+) M(H_-),$$

where H_+ and H_- are the left and right "halves" of H obtained by removing the edges of H that cross P, with the additional specification that all edges of H_+ along ℓ are given weight 1/2.

However, one readily sees that H_+ is isomorphic to $T(G'_{\mathrm{L}})\backslash v$, where G'_{L} is the graph obtained from G_{L} by weighting its edges along ℓ by 1/2. Similarly, H_- is seen to be isomorphic to $T(G_2)\backslash u$, where u is the vertex of G_2 obtained by identifying all vertices of G_{R} that are on ℓ. Therefore, by Lemma 5, we have $M(H_+) = t(G'_{\mathrm{L}})$ and $M(H_-) = t(G_2)$. Moreover, it is easy to see that $w(H) = v_\ell - 1$, where v_ℓ is the number of vertices of G on ℓ. Thus, the above displayed equation can be rewritten as

$$t(G) = 2^{v_\ell - 1} t(G'_{\mathrm{L}}) t(G_2).$$

To prove the statement of the theorem it suffices to show that

$$2^{v_\ell - 1} t(G'_{\mathrm{L}}) = 2^{\omega(G)} t(G_1).$$

It follows from the definitions of $\omega(G)$ and v_ℓ that $\omega(G) = v_\ell - 1 - e_\ell$, where e_ℓ is the number of edges of G along ℓ. Therefore the last equation amounts to

$$2^{e_\ell} t(G'_{\mathrm{L}}) = t(G_1).$$

Given the definitions of G'_{L} and G_1, this follows by repeated application of Lemma 6.

Lemma 6. *Let G be a graph with vertex set V and edge set E. Let a, b and x be three distinct points outside V, and let* $\{c, d\} \in E$. *Construct the graph* $G_1 = (V_1, E_1)$ *by setting* $V_1 = V \cup \{a, b\}$, $E_1 = E \cup \{a, b\} \cup \{a, c\} \cup \{b, d\}$. *Assign weight* 1 *to all edges of* G_1 *except* $\{a, b\}$; *weight* $\{a, b\}$ *by* $1/2$. *Let* $G_2 = (V_2, E_2)$ *be the graph with* $V_2 = V \cup \{a, b, x\}$ *and* $E_2 = E \cup \{a, x\} \cup \{b, x\} \cup \{a, c\} \cup \{b, d\}$. *Weight all edges of* G_2 *by* 1. *Then we have*

$$2t(G_1) = t(G_2).$$

Proof. Partition the family $\mathcal{T}(G_1)$ of the spanning trees of G_1 as $C_1 \cup C_2$, where C_1 consists of the spanning trees of G_1 that contain the edge $\{a, b\}$ and C_2 of the spanning trees not containing this edge. Write $\mathcal{T}(G_2) = C'_1 \cup C'_2 \cup C'_3$, where C'_1 is the collection of spanning trees of G_2 that contain both $\{a, x\}$ and $\{b, x\}$, C'_2 consists of the spanning trees not containing $\{a, x\}$, and C'_3 of those not containing $\{b, x\}$.

Contracting the edge $\{b, x\}$ to a point defines a bijection $g : C'_2 \mapsto C_2$. Similarly, contracting the edge $\{a, x\}$ to a point defines a bijection $h : C'_3 \mapsto C_2$. Removing x and the incident edges and including the edge $\{a, b\}$ defines a bijection $f : C'_1 \mapsto C_1$. Furthermore, for any spanning tree T of G_2 the weight $wt(T)$ of T and that of its image satisfy $wt(f(T)) = \frac{1}{2} wt(T)$ and $wt(g(T)) = wt(h(T)) = wt(T)$. This implies the statement of the lemma. $\square$

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